Complex Numbers Exercises: Solutions

1. write in the form $x + iy$:

(a). $\dfrac{1}{2+2i}$

$$\frac{1}{2+2i} = \frac{1}{2+2i} \cdot \frac{2-2i}{2-2i} = \frac{2-2i}{4+4} = \tfrac{1}{4} - \tfrac{1}{4}i.$$

(b). $\dfrac{1}{i^3}$

$$\frac{1}{i^3} = \frac{1}{-i} = \frac{-1}{i} \cdot \frac{i}{i} = \frac{-i}{-1} = i.$$

(c). $i(1+i)(1-i)^2$

$$i(1+i)(1-i)^2 = (i-1)(1-i)^2 = (i-1)(1-2i+i^2) = (i-1)(-2i) = -2i^2 + 2i = 2 + 2i.$$

2. Write in the polar and the exponential polar form:

(a). $\dfrac{1}{2+2i}$

(see 1.a.) $= \frac{1}{4} - \frac{1}{4}i$, so, $|z| = \sqrt{\frac{1}{16} + \frac{1}{16}} = \frac{\sqrt{2}}{4} = \frac{1}{4}\sqrt{2}$. From drawing $\frac{1}{4} - \frac{1}{4}i$ we see directly that $\varphi = -\frac{1}{4}\pi$, so

$$\tfrac{1}{4} - \tfrac{1}{4}i = \tfrac{1}{4}\sqrt{2}\left(\cos\left(-\tfrac{\pi}{4}\right) + i\sin\left(-\tfrac{\pi}{4}\right)\right) = \tfrac{1}{4}\sqrt{2}e^{-i\frac{\pi}{4}}.$$

(b). $-1 + i\sqrt{3}$

$|z| = \sqrt{1+3} = 2$. By drawing and using an $\arctan$ we find $\varphi = \frac{2}{3}\pi$. So $-1 + i\sqrt{3} = 2\left(\cos\left(\frac{2}{3}\pi\right) + i\sin\left(\frac{2}{3}\pi\right)\right) = 2e^{i\frac{2}{3}\pi}$.

(c). $\sqrt{1+i}$

First look at $1+i$: this has modulus $\sqrt{2}$ and argument $\frac{\pi}{4}$, so $1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$. Now take the square root:

$$(1+i)^{1/2} = \left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^{1/2} = 2^{\frac{1}{4}}e^{i\frac{\pi}{8}} = \sqrt[4]{2}\left(\cos\left(\tfrac{\pi}{8}\right) + i\sin\left(\tfrac{\pi}{8}\right)\right).$$

3. Give all roots (solutions) of $z^2 + z + 1 = 0$.

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1-4}}{2} = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{-3} = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3}i.$$

4. Split into factors: $z^2 + 1$.

Solutions of $z^2 + 1 = 0$ are $z = \pm i$. (You can see this directly from $z^2 = -1$ or by using the quadratic *abc*-formula.) So: $z^2 + 1 = (z - i)(z + i)$.

5. Multiplying a complex z by i is the equivalent of rotating z in the complex plane by $\pi/2$.
(a). Verify this for $z = 2 + 2i$
(b). Verify this for $z = 4 - 3i$
(c). Show that $zi \perp z$ for all complex z.

The easiest way is to use linear algebra: set $z = x + iy$. Then $zi = ix - y$. This corresponds to the vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} -y \\ x \end{pmatrix}$ in the complex plane respectively. Since the dot product of these vectors is 0, they are perpendicular.

6. Calculate Im $((i + 1)^8 z^2)$ for $z = x + iy$.

$$i + 1 = \sqrt{2}\left(\cos\left(\tfrac{\pi}{4}\right) + i\sin\left(\tfrac{\pi}{4}\right)\right)$$
$$(i + 1)^8 = 2^4\left(\cos\left(2\pi\right) + i\sin\left(2\pi\right)\right) = 16(1 + i \cdot 0) = 16$$

$$z^2 = x^2 + 2ixy - y^2$$
$$(i + 1)^8 z^2 = 16(x^2 - y^2) + 32ixy$$
$$\text{Im}\,((i + 1)^8 z^2) = 32xy.$$

7. Find an expression for $\sin(3\theta)$ in terms of $\sin(\theta), \cos(\theta)$.

By de Moivre's formula (using shorthands $\cos\theta = c$ and $\sin\theta = s$):

$(c + is)^3 = \cos(3\theta) + i\sin(3\theta)$
working out the left side gives:
$(c + is)(c^2 + 2ics + -s^2) =$
$c^3 + 2ic^2 s - cs^2 + isc^2 - 2cs^2 - is^3 =$
$(c^3 - cs^2 - 2cs^2) + i(2c^2 s + sc^2 - s^3) =$
$(c^3 - 3cs^2) + i(3c^2 s - s^3).$

So $\cos(3\theta) + i\sin(3\theta) = (c^3 - 3cs^2) + i(3c^2s - s^3)$. This equality only holds if both the real and the imaginary parts of the equation hold. In this case, we are only interested in the imaginary part, because this equals $\sin(3\theta)$, so:

$$\sin(3\theta) = 3\cos^2(\theta)\sin(\theta) - \sin^3(\theta).$$

8.(advanced) Solve $z^4 + 16 = 0$ for complex z, then use your answer to factor $z^4 + 16$ into two factors with real coefficients.

Alternative 1: write the equation as $(z^2)^2 + 16 = 0$:

$$\begin{array}{lll}
(z^2)^2 = -16 & & \\
z^2 = 4i & \vee & z^2 = -4i \\
z^2 = 4e^{i\frac{\pi}{2}} & \vee & z^2 = 4e^{i\frac{-\pi}{2}} \\
z = \pm 2e^{i\frac{\pi}{4}} & \vee & z = \pm 2e^{i\frac{-\pi}{4}} \\
z = \pm(2(\frac{1}{2}\sqrt{2} + i\frac{1}{2}\sqrt{2})) & \vee & z = \pm(2(\frac{1}{2}\sqrt{2} - i\frac{1}{2}\sqrt{2})) \\
z = \pm(\sqrt{2} + i\sqrt{2}) & \vee & z = \pm(\sqrt{2} - i\sqrt{2})
\end{array}$$

So

$$z^4 + 16 = (z - (\sqrt{2} + i\sqrt{2}))(z - (-\sqrt{2} - i\sqrt{2}))(z - (\sqrt{2} - i\sqrt{2}))(z - (-\sqrt{2} + i\sqrt{2})).$$

We can reduce these four factors to two factors by (e.g.) multiplying factors 1&2 and 3&4, but this leads to $z^4 + 16 = (z^2 - 4i)(z^2 + 4i)$, i.e. not with real coefficients. Multiplying factors 1&3 and 2&4 gives the desired answer:

$$z^4 + 16 = (z^2 - 2\sqrt{2}z + 4)(z^2 + 2\sqrt{2}z + 4).$$

Alternative 2:

$z^4 + 16 = 0$
$z^4 = -16 = 16(\cos(\pi) + i\sin(\pi)) = 16e^{i\pi}$
more general:
$z^4 = 16e^{i(\pi + k2\pi)}$ with k integer (this is necessary to find all roots with a valid argument).
$z = \sqrt[4]{16}e^{i(\frac{\pi}{4} + k\frac{\pi}{2})}$
$z = 2e^{i(\frac{\pi}{4} + k\frac{\pi}{2})}$

For valid arguments ($-\pi < \varphi \leq \pi$) this yields $z = 2e^{-\frac{3}{4}\pi}, 2e^{-\frac{1}{4}\pi}, 2e^{\frac{1}{4}\pi}, 2e^{\frac{3}{4}\pi}$, which leads to the same solutions as in alternative 1. (The rest; the factorization is the same as in alternative 1.)